**Géométrie Lipschitz des singularités**

**Lipschitz Geometry of Singularities**

*Marseille, 7-10 november 2017*

*Program*

## Courses

**Immanuel Halupczok** *3 one-hour lectures*

*Lipschitz stratifications*

*Abstract.* To understand the singularities of a set X (e.g. of an algebraic subset of $\mathbb{R}^n$), one can use stratifications: partitions of X (into sets called "strata") according to "how singular" the points are. The most classical notion of stratification is that of a Whitney stratification. In this mini-course, I will describe stronger stratifications, namely the Lipschitz stratifications introduced by Mostowski. Lipschitz stratifications are mysterious for several reasons: On the one hand, the usual definition looks tremendously technical. On the other hand, it also turns out that Lipschitz stratifications are often forced to have very uncanonical strata. The main aim of my course is to lift these mysteries, i.e., to persuade the audience that the definition is "the right one" and also to explain where these uncanonical strata come from. To this end, I will take a modern approach which consists in working in a larger field which contains infinitesimal elements (which can e.g. be used to simplify the definition of Lipschitz stratifications). Most likely, I will also find a bit of time to say something about the proof of existence of Lipschitz stratifications.

**Walter Neumann** *3 one-hour lectures*

*Lipschitz geometry of complex surface germs*

*Abstract.* We will describe analytic invariants which are determined by the outer Lipschitz geometry of surface germs, including the multiplicity.

**Laurenciu Paunescu** *3 one-hour lectures*

*Arc-wise analytic equisingularity*

*Abstract.* I will give a detailed course on our recent result on Whitney conjecture (A. Parusinski and L. Paunescu, Advances in Math, Vol. 309, 254-305, https://arxiv.org/pdf/1503.00130.pdf), in particular on Whitney interpolation, Arc-wise analytic stratification, Whitney fibering conjecture and Zariski equisingularity.

## Talks

**Leonardo Meireles Camara** *30 mn*

*Analytic and bi-Lipschitz moduli of quasi-homogeneous functions*

*Abstract.* We present a complete set of invariants determining the analytic type of reduced quasi-homogeneous functions at $(\mathbb{C}^2, 0)$ with respect to $\mathcal{R}$-equivalence.

We describe the relationship between these invariants and the Henry-Parusiński invariants related to bi-Lipschitz $\mathcal{R}$-equivalence at $(\mathbb{C}^2, 0)$. We show that any non-degenerate continuous family of reduced quasi-homogeneous functions with constant Henry-Parusiński invariant is analytically trivial, generalizing a result due to Fernandes and Ruas concerning strongly bi-Lipschtiz families. Furthermore, we show that there are only a finite number of distinct bi-Lipschitz classes between quasi-homogeneous functions with the same Henry-Parusiński invariant providing a maximum quota for this number.

**Alexandre Fernandes** *50 mn*

*Multiplicity of analytic surface singularities in $\mathbb{C}^3$ is bi-Lipschitz invariant.*

*Abstract.* We shall present a proof that the multiplicity of complex analytic surface (not necessarily isolated) singularities in $\mathbb{C}^3$ is a bi-Lipschitz invariant.

**Rodrigo Mendes** *30 mn*

*Lipschitz contact equivalence and real analytic functions germs in the plane*

*Abstract.* In [2], the authors construct a complete invariant of the two variable definable (in a polynomially bounded o-minimal structure) functions germs with respect to the Lipschitz contact equivalence. The invariant is called a pizza of the function. This set is formed by the sequence of triangles equipped with width functions along to the segments in $\mathbb{Q} \cup \{\infty\}$. Motivated by some open questions in this paper, we consider the pizza of real analytic functions germs. We prove that the "analytic pizzas" are glued along to the boundary of your pieces. More precisely, the width function defined in all the segments of the pizza is a continuous function. Moreover, we obtain that the slope of the width functions in the corresponding pieces is always positive. Our construction is considered from the zones in the set of arcs. We will discuss generalizations and consequences of this results. This is a joint work with Lev Birbrair.

*References*
[1] L. Birbrair. J. C. F. Costa, A. Fernandes, M.A.S. Ruas, K-bi-Lipschitz equivalence of real function-germs. Proc. Amer. Math. Soc. 135 (2007), no. 4, 1089-1095.
[2] L. Birbrair, A. Fernandes, A. Gabrielov, V. Grandjean, Lipschitz contact equivalence of function germs in $\mathbb{R}^2$, (2014), arxiv:1406.2559
[3] S. Koike, A, Parusinski, Equivalence relations for two variable real analytic function germs. J. Math. Soc. Japan 65 (2013), no. 1, 237-276.
[4] S. Koike and A. Parusinski Blow-analytic equivalence of two variable real analytic function germs. J. Algebraic Geom. 19 (2010), no. 3, 439-472.
[5] L. Birbrair, A. Fernandes, T. Gaffney, V. Grandjean, Blow analytic equivalence versus contact-bi- Lipschitz equivalence, (2016), arxiv:1601.06056.
[6] L. Birbrair, J. C. F. Costa, R. Mendes, E. Sena, Finiteness theorem for multi-K-Bi-Lipschitz equivalence of map germs. arxiv:1706.08156.

**Françoise Michel** *50 mn*

*On the behaviour of Hironaka quotients in good resolutions*

*Abstract.* Let $(X, p)$ be a complex analytic normal surface germ. We consider a finite analytic morphism $\phi = (f, g) : (X, p) \longrightarrow (\mathbb{C}^2, 0)$ where $f$ and $g$ are two

analytic function germs defined on $(X,p)$. Let $G(Y)$ be the dual graph of a good resolution $\rho : (Y, E_Y) \to (X,p)$ of $\phi$. The morphism $\phi = (f,g)$ associates a quotient to each vertex of $G(Y)$. We call them Hironaka quotients (or contact quotients) of $\phi = (f,g)$.

In a recent work with H.Maugendre [3], we show that there exist oriented paths in $G(Y)$ along which the Hironaka quotients of $(f,g)$ increase. They are constant for the vertices which belong to the closure of a connected component of the complement of the union of the oriented paths. The proof is based on the Hirzebruch-Jung method to obtained good resolutions of $\phi$.

In the talk we will define and describe the behaviour of Hironaka quotients. We will show how they are related to the discriminant curve. We can establish relations between Hironaka quotients and the polar curves involved in the works of Birbrair-Neumann-Pichon and Neumann-Pichon on the Lipschitz geometry of complex surfaces. We will give an example which illustrated this point.

The talk will be based on the following works: in [1], D.T. Lê, H. Maugendre, and C. Weber use Hironaka quotients to prove that the first Puiseux pairs of the discriminant of $\phi = (f,g)$ are topological invariants of $(X,(f,g))$. In [2], the behaviour of the strict transform of the discriminant by a good minimal resolution $\rho : (Y, E_Y) \to (X,p)$ is described with the help of the Hironaka quotients of $(f,g)$.

*References*

[1] D.T. Lê, H. Maugendre, C. Weber, *Geometry of critical loci*, Journal of the L.M.S. 63 (2001), 533-552.

[2] F. Michel, *Jacobian curves for normal complex surfaces*, Brasselet, J-P. (ed.) et al., Singularities II. Geometric and topological aspects. Proceedings of the international conference "School and workshop on the geometry and topology of singularities" in honor of the 60th birthday of Lê Dũng Tràng, Cuernavaca, Mexico, January 8-26, 2007. Providence, RI: American Mathematical Society (AMS). Contemporary Mathematics 475, 135-150 (2008).

[3] H. Maugendre, F. Michel, On the growth behaviour of Hironaka quotients, arXiv:1707.02219 (July 2017).

**Hussein Mourtada** *50 mn*

*A polyhedral criterion for splitting of valuations*

*Abstract.* We will give an algorithmic criterion for splitting of valuations in algebraic extensions, based on Newton polyhedra. This is a work in progress with Dale Cutkosky and Bernard Teissier

**Claudio Murolo** *50 mn*

*Smooth Whitney fibering conjecture II; Main details of the proofs.*

*Abstract.* We improve upon the first Thom-Mather isotopy theorem for Whitney stratified sets. In particular, for the more general Bekka stratified sets we show that there is a local foliated structure with continuously varying tangent spaces, thus proving the smooth version of the Whitney fibering conjecture. A regular wing structure is also shown to exist locally, for both Whitney and Bekka stratifications. The proofs involve integrating carefully chosen controlled distributions of vector fields. As an application of our main theorem we show the density of the subset

of strongly topologically stable mappings in the space of all smooth quasi-proper mappings between smooth manifolds, an improvement of a theorem of Mather.

This is the second talk of a series of two. The first talk is given by David Trotman.

**Nhan Nguyen** *30 mn*

*Classification of Lipschitz simple function germs*

*Abstract.* The classification of smooth function germs w.r.t the smooth right equivalent is well-known by the works of Arnod in the 1960s. The simplest corank 2 function germs that admit smooth modality is $X_9$ and $J_{10}$. Henry and Parusinski proved a famous result that the smooth one-modal $J_{10}$ that has the normal form $f_t(x, y) = x^3 - t^2xy + y^6$ is also Lipschitz modal, i.e. if $t_1 \neq t_2$ then $f_{t_1}$ and $f_{t_2}$ are not bi- Lipschitz right equivalent. This encourages us to find a clasification for smooth function germs w.r.t bi-Lipschitz right equivalent. In this paper, we introduce the notion of Lipschitz simple function germs and set out to classify them under bi-Lipschitz right equivalent. The method to do it includes two steps, first we show that rank and corank are Lipschitz invariants, this helps in writing up the normal forms; second we check if a germ where smooth modality occurs is Lipschitz simple or not. This is a joint work with M. Ruas and S. Trivedi.

**Krzystof Jan Nowak** *50 mn*

*The closeness theorem over henselian valued fields and its applications*

*Abstract.* The aim of this talk is to develop geometry of algebraic subvarieties of $K^n$ over arbitrary Henselian valued fields $K$ of equicharacteristic zero. This is a continuation of my previous article, devoted to algebraic geometry over rank one valued fields, which in general requires more involved techniques and to some extent new treatment. Again, at the center of my approach is the closedness theorem that the projections $K^n \times \mathbb{P}^m(K) \to K^n$ are definably closed maps. Hence we obtain a descent property for blow-ups, which enables applications of resolution of singularities in much the same way as over the locally compact ground field. As before, the proof of that theorem uses i.a. certain cell decomposition, the local behaviour of definable functions of one variable and fiber shrinking, a relaxed version of curve selection. But now, to achieve the former result, I first examine functions given by algebraic power series. The results established include: several versions of curve selection and of the Lojasiewicz inequality, piecewise continuity of definable functions and Hölder continuity of functions on closed bounded subsets of $K^n$, extending continuous hereditarily rational functions and the theory of regulous functions, sets and sheaves. Two basic tools applied are quantifier elimination for Henselian valued fields due to Pas and relative quantifier elimination for ordered abelian groups (in a many-sorted language with imaginary auxiliary sorts) due to Cluckers-Halupczok.

**Helge Pedersen** *30 mn*

*Lipschitz normal embeddings in the space of matrices.*

*Abstract.* The germ of an algebraic variety is naturally equipped with two different metrics up to bilipschitz equivalence. The inner metric and the outer metric. One

calls a germ of a variety Lipschitz normally embedded if the two metrics are bilipschitz equivalent. In this talk we prove Lipschitz normal embeddedness of some algebraic subsets of the space of matrices. These include the space of matrices, symmetric matrices and skew-symmetric matrices of rank equal to a given number and their closures, the upper triangular matrices with determinant 0 and linear space transverse to the rank stratification away from the origin.

**Edson Sampaio** *50 mn*

*Degree of complex algebraic sets as a bi-Lipschitz invariant*

*Abstract.* In this talk we address a metric version of Zariski's multiplicity conjecture at infinity that says that two complex algebraic affine sets which are bi-Lipschitz homeomorphic at infinity must have the same degree. More specifically, we prove that relative multiplicities at infinity of complex algebraic sets in $\mathbb{C}^n$ are invariant under bi-Lipschitz homeomorphisms at infinity, we also show that the local metric version of Zariski's multiplicity conjecture and that one at infinity are equivalent and we get a proof that degree of complex algebraic surfaces in $\mathbb{C}^3$ is invariant of the bi-Lipschitz equivalence at infinity. This is a joint work with Alexandre Fernandes.

**Saurabh Trivedi** *30 mn*

*Bi-Lipschitz geometry of contact orbits in the boundary of the nice dimensions*

*Abstract.* (Joint work with Maria Ruas) We give a complete description of the Thom-Mather stratification in the boundary of the nice dimensions. This requires a classification of contact unimodular stratum in all pairs of dimensions lying in the boundary of the nice dimensions where smoothly stable maps are not dense. We then show that this stratification is contact bi-Lipschitz invariant. Some conjectures will also be mentioned.

**Mihai Tibar** *30 mn*

*Lipschitz invariants of 2-variables complex functions*

*Abstract.* I will introduce some analytic invariants which candidate as Lipschitz invariants. Joint work in progress with Laurentiu Paunescu.

**David Trotman** *50 mn*

*Smooth Whitney fibering conjecture I. Background, statements and applications.*

*Abstract.* We improve upon the first Thom-Mather isotopy theorem for Whitney stratified sets. In particular, for the more general Bekka stratified sets we show that there is a local foliated structure with continuously varying tangent spaces, thus proving the smooth version of the Whitney fibering conjecture. A regular wing structure is also shown to exist locally, for both Whitney and Bekka stratifications. The proofs involve integrating carefully chosen controlled distributions of vector fields. As an application of our main theorem we show the density of the subset of strongly topologically stable mappings in the space of all smooth quasi-proper mappings between smooth manifolds, an improvement of a theorem of Mather.

The second talk will be given by Claudio Murolo.

## Posters

**Hellen Monção de Carvalho Santana**

*L'obstruction d'Euler, le nombre de Brasselet et les cycles de Lê*

*Abstract.* L'obstruction locale a été définie par MacPherson comme un outil pour l'étude des classes caractéristiques des variétés singulières. Brasselet, Massey, Parameswaran et Seade ont présenté une généralisation de ce concept en ajoutant les informations d'une fonction $f$ avec une singularité isolée définie sur une variété singulière, appelée l'obstruction d'Euler d'une fonction $f$. Plus récemment, Dutertre et Grulha ont présenté une autre généralisation de l'obstruction locale, appelée nombre de Brasselet, qui est bien définie même si $f$ a une singularité non-isolée. Dans ce travail, nous présentons quelques questions sur l'obstruction d'Euler locale et ses généralisations. L'objectif est de chercher des relations entre le nombre de Brasselet et d'autres invariants, comme le nombre de Lê, de chercher des formules pour simplifier ces calculs puis, avec ces relations, d'évaluer les implications sur la topologie de fonctions définies sur des variétiés singulières.